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THE CONSTRUCTION OF VARIATIONAL PRINCIPLES†

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A method of constructing variational principles (VPs) for a class of problems in mechanics is presented. The VPs are derived from variational problems equivalent to satisfying constitutive relations. The physical side of such a derivation scheme consists in setting up variational problems by considering the minimum rate of energy accumulation and dissipation. In doing so one can distinguish the mechanisms of energy accumulation and dissipation, which define the number of variables in the VP. The VPs are constructed for a system of transfer equations in the steady case and for the problem of the seepage of an incompressible fluid in a deformable medium of complex rheology. The proposed approach simplifies the procedure for constructing dual VPs and can be used to construct other VPs for media with complex rheology. The derivation of the VP remains the same for problems whose solution is defined by the minimum of potential energy. © 1997 Elsevier Science Ltd. All rights reserved.

1. THE CONSTRUCTION OF VARIATIONAL PRINCIPLES (VPs)

For many problems in the mechanics of continuous media the VP can be written in the form

$$\inf_{\mathbf{c}\in M} I_1(\mathbf{Y}) = \inf_{\mathbf{c}\in M}\left[\int_\Omega (\varphi(\mathbf{Y}) + f(\mathbf{c}))d\Omega + \int_\Gamma F(\mathbf{c})d\Gamma\right] \tag{1.1}$$

where $\varphi(\mathbf{Y})$ is a smooth convex functional, $f(\mathbf{c})$ and $F(\mathbf{c})$ are linear functionals of the components of the vector $\mathbf{c}$, $\mathbf{Y} = \mathbf{Y}(\mathbf{c})$, $\mathbf{c} = \mathbf{c}(\Omega)$ (for example, $\mathbf{c}$ is the displacement vector, $\mathbf{Y}(\mathbf{c})$ is the strain tensor, c is the pressure and $\mathbf{Y}(\mathbf{c})$ is the pressure gradient), in particular, $\mathbf{Y} = \mathbf{c}(\Omega)$, Ω is the solution domain and Γ is the boundary of Ω. Furthermore, suppose that a solution and the VP (1.1) exist for the specified boundary-value problem. Conditions for a solution of the problem to exist are not discussed in this paper, nor is the question of uniqueness. In the VP (1.1) one needs to establish the form of the functionals $\varphi(\mathbf{Y})$, $f(\mathbf{c})$ and $F(\mathbf{c})$ and the set of constraints imposed on $\mathbf{c}$.

We introduce the following notation $(\cdot) = (\cdot)^\circ$, $\mathbf{c} = \mathbf{c}^\circ$, $\mathbf{Y} = \mathbf{Y}(\mathbf{c}^\circ) = \mathbf{Y}^\circ$ for the variables taken on the solution. With (1.1) we associate the variational problem

$$\inf_{\mathbf{Y}} B_1^\circ(\mathbf{Y}) = \inf_{\mathbf{Y}} \int_\Omega [\varphi(\mathbf{Y}) - \mathbf{X}^\circ\mathbf{Y}]d\Omega \tag{1.2}$$

which is equivalent to $\mathbf{X}^\circ = \operatorname{grad}\varphi(\mathbf{Y}^\circ)$ [1, 2], where $\mathbf{XY} = \mathbf{X}_i\mathbf{Y}_i = \mathbf{X}_1\mathbf{Y}_1 + \ldots + \mathbf{X}_k\mathbf{Y}_k$ The vector $\mathbf{Y}^\circ$ is a solution of (1.2) and $\mathbf{c}^\circ$ is a solution of (1.1) and (1.2). Problem (1.2) is trivial, since $\mathbf{X}^\circ$ must be known in the whole solution domain Ω to determine $\mathbf{Y}^\circ$.

To construct a full-valued VP it is necessary to transform (1.2) into the form (1.1). The transformations of (1.2) are admissible if the solutions of the variational problems related by the transformations are attained for the same field $\mathbf{c}^\circ$. A similar variational problem

$$\inf_{\mathbf{X}} B_2^\circ(\mathbf{X}) = \inf_{\mathbf{X}} \int_\Omega [\varphi^*(\mathbf{Y}) - \mathbf{X}\mathbf{Y}^\circ]d\Omega \tag{1.3}$$

can be written for the construction of the VP

$$\inf_{\mathbf{b}\in M^*} I_2(\mathbf{X}) = \inf_{\mathbf{b}\in M^*}\left[\int_\Omega (\varphi^*(\mathbf{X}) + f^*(\mathbf{b}))d\Omega + \int_\Gamma F^*(\mathbf{b})d\Gamma\right] \tag{1.4}$$

dual to (1.1), where $\mathbf{X} = \mathbf{X}(\mathbf{b})$, $\mathbf{b} = \mathbf{b}(\Omega)$ and $\varphi^*(\mathbf{X})$ is the adjoint functional related to $\varphi(\mathbf{Y})$ by the Young–Fenchel transformation [3]

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$$\varphi^*(\mathbf{X}) = \sup_{\mathbf{Y}}[\mathbf{X}\mathbf{Y} - \varphi(\mathbf{Y})]$$

Depending on the nature of the problem the functional $\varphi(\mathbf{Y})$ in (1.1) can be chosen by considering the minimum of the potential energy or the minimum rate of energy accumulation and dissipation. By means of $\varphi(\mathbf{Y})$, using the Young–Fenchel transformation with respect to some of the variables $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_k)$, one can define the partially adjoint functionals

$$\varphi(\mathbf{Y}_{0m}, \mathbf{X}_{mk}) = \sup_{\mathbf{Y}_{mk}}[\mathbf{X}_{mk}\mathbf{Y}_{mk} - \varphi(\mathbf{Y})]$$

where $\mathbf{Y}_{0m} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_m)$, $\mathbf{Y}_{mk} = (\mathbf{Y}_{m+1}, \ldots, \mathbf{Y}_k)$. Then the variational problem for the construction of the VP dual with respect to some of the variables will have the form

$$\inf_{\mathbf{Y}_{0m}} \sup_{\mathbf{X}_{mk}} B_3^\circ(\mathbf{Y}_{0m}, \mathbf{X}_{mk}) = \inf_{\mathbf{Y}_{0m}} \sup_{\mathbf{X}_{mk}} \int_\Omega [-\varphi(\mathbf{Y}_{0m}, \mathbf{X}_{mk}) - \mathbf{X}_{0m}^\circ \mathbf{Y}_{0m} + \mathbf{X}_{mk}\mathbf{Y}_{mk}^\circ] d\Omega \tag{1.5}$$

Instead of (1.2), (1.3) and (1.5), in the construction of the VP one can begin with the variations

$$\delta B_1^\circ(\mathbf{Y}) = \int_\Omega [\delta\varphi(\mathbf{Y}) - \mathbf{X}^\circ \delta\mathbf{Y}] d\Omega, \quad \delta B_2^\circ(\mathbf{X}) = \int_\Omega [\delta\varphi^*(\mathbf{X}) - \mathbf{Y}^\circ \delta\mathbf{X}] d\Omega$$

$$\delta B_3^\circ(\mathbf{Y}_{0m}, \mathbf{X}_{mk}) = \int_\Omega [-\delta\varphi(\mathbf{Y}_{0m}, \mathbf{X}_{mk}) - \mathbf{X}_{0m}^\circ \delta\mathbf{Y}_{0m} + \mathbf{Y}_{mk}^\circ \delta\mathbf{X}_{mk}] d\Omega$$

which are equal to zero

$$\delta B_1^\circ(\mathbf{Y}) = 0, \quad \delta B_2^\circ(\mathbf{X}) = 0, \quad \delta B_3^\circ(\mathbf{Y}_{0m}, \mathbf{X}_{mk}) = 0 \tag{1.6}$$

if and only if the constitutive relations between $\mathbf{X}$ and $\mathbf{Y}$ are satisfied.

Variational problems similar to (1.2), (1.3), (1.5) and (1.6) can be written for any smooth convex functionals relating arbitrary dual variables $\mathbf{X}$ and $\mathbf{Y}$ by $\mathbf{X} = \operatorname{grad} \varphi(\mathbf{Y})$ or $\mathbf{Y} = \operatorname{grad} \varphi^*(\mathbf{X})$, and they can be used to construct the VP.

The above assertions hold for subgradient relations between $\mathbf{X}$ and $\mathbf{Y}$ [2]

$$\mathbf{X} \in \partial\varphi(\mathbf{Y}), \quad \mathbf{Y} \in \partial\varphi^*(\mathbf{X})$$

where $\varphi(\mathbf{Y})$ is a convex lower semicontinuous characteristic functional, $\mathbf{X}$ is the subgradient of $\varphi(\mathbf{Y})$ at $\mathbf{Y}$ and $\partial\varphi(\mathbf{Y})$ is the set of all subgradients of $\varphi(\mathbf{Y})$ at $\mathbf{Y}$ consisting of one element $\operatorname{grad}\varphi(\mathbf{Y})$ in the case when $\varphi(\mathbf{Y})$ is smooth. The results can be extended to problems with constitutive relations of non-potential form, the which variational equations can be constructed.

2. VARIATIONAL PRINCIPLES FOR THE TRANSFER EQUATIONS

We will construct the VP for a system of transfer equations in the steady case

$$\operatorname{div} \mathbf{J}_i = \sum_{j=1}^{k} Q_{ji} + \sigma_i^*, \quad i = 1, 2, \ldots k \tag{2.1}$$

$$\partial\Psi(\mathbf{I}) / \partial\mathbf{J}_i = -\nabla p_i, \quad \partial\Psi(\mathbf{I}) / \partial Q_{ji} = p_j - p_i \tag{2.2}$$

or

$$\partial\Phi(\mathbf{P}) / \partial\nabla p_i = -\mathbf{J}_i, \quad \partial\Phi(\mathbf{P}) / \partial(p_j - p_i) = Q_{ji}$$

where $\Psi(\mathbf{I})$ is a dissipative potential, $\mathbf{I} = (\mathbf{J}_1, \ldots, \mathbf{J}_k, Q_{21}, \ldots, Q_{k1}, Q_{32}, \ldots, Q_{k2}, \ldots, Q_{k,k-1})$ are the thermodynamic flows, Q_{ij} are the densities of internal sources ($Q_{ij}, = Q_{ji}, Q_{ii} = 0$), σ^*_i are the given densities of external sources, $\Phi(\mathbf{P}) = \sup p_1 |-\mathbf{J}_i \nabla p_i + {}^1\!/_2 Q_{ji}(p_j - p_i)$ is the adjoint dissipative potential and $\mathbf{P} = (-\nabla p_1, \ldots, \nabla p_k, p_2 - p_1, \ldots, p_k - p_1, p_3 - p_2, \ldots, p_k - p_2, \ldots, p_k - p_{k-1})$ are the thermodynamic forces. In the case of unconnected dissipation mechanisms [4]

$$\Psi(\mathbf{I}) = \sum_{i=1}^{k} \Psi_i + \sum_{j>i}^{k} \Psi_{ji}, \quad \Psi_i = \Psi_i(\mathbf{J}_i), \quad \Psi_{ji} = \Psi_{ji}(Q_{ji})$$

$$\Phi(\mathbf{P})=\sum_{i=1}^{k}\Phi_i+\sum_{j>i}^{k}\Phi_{ji},\quad \Phi_i=\Phi_i(\nabla p_i),\quad \Phi_{ji}=\Phi_{ji}(p_j-p_i)$$

we have

$$\partial\Psi_i/\partial\mathbf{J}_i=-\nabla p_i,\quad \partial\Psi_{ji}/\partial Q_{ji}=p_j-p_i \tag{2.3}$$

Here and henceforth, when constructing specific VPs the potentials will be assumed smooth and convex.

Considering the minimum rate of energy dissipation, to construct the VP in terms of the variables **I** we shall begin with the variational problem (1.2), where $\varphi(\mathbf{Y})=\Psi(\mathbf{I})$ and $\mathbf{Y}=\mathbf{c}=\mathbf{I}$. We transform (1.2) using (2.1) and (2.2) into a VP in which the value of $\mathbf{P}^\circ$ is not required in the whole domain Ω

$$\inf_{\mathbf{I}}\int_\Omega(\Psi(\mathbf{I})-\mathbf{P}^\circ\mathbf{I})d\Omega=\inf_{\mathbf{I}}\int_\Omega(\Psi(\mathbf{I})+\mathbf{J}_i\nabla p_i^\circ-\sum_{j>i}^{k}Q_{ji}(p_j^\circ-p_i^\circ))d\Omega=$$

$$=\inf_{\mathbf{I}}\left[\int_\Omega(\Psi(\mathbf{I})-p_i^\circ\operatorname{div}\mathbf{J}_i-\sum_{j>i}^{k}Q_{ji}(p_j^\circ-p_i^\circ))d\Omega+\int_\Gamma J_{in}p_i^\circ d\Gamma\right]=$$

$$=\inf_{\mathbf{I}}\left[\int_\Omega\left(\Psi(\mathbf{I})-p_i^\circ\left(\operatorname{div}\mathbf{J}_i-\sum_{j=i}^{k}Q_{ji}\right)\right)d\Omega+\int_\Gamma J_{in}p_i^\circ d\Gamma\right]=$$

$$=\inf_{\mathbf{I}\in(2.1)}\left[\int_\Omega\Psi(\mathbf{I})d\Omega+\int_\Gamma J_{in}p_i^\circ d\Gamma\right]-\int_\Omega p_i^\circ\sigma_i^* d\Omega$$

Formulating the boundary conditions

$$J_{in}=J_{in}^\circ\quad\text{on }\Gamma_{iq},\quad \Gamma_{iq}+\Gamma_{ip}=\Gamma,\quad i=1,2,\ldots,k \tag{2.4}$$

and discarding the constant terms, we obtain the VP

$$\inf_{\mathbf{I}\in(2.1),(2.4)}I_1(\mathbf{I})=\inf_{\mathbf{I}\in(2.1),(2.4)}\left[\int_\Omega\Psi(\mathbf{I})d\Omega+\sum_{i=1}^{k}\int_{\Gamma_{ip}}J_{in}p_i^\circ d\Gamma\right] \tag{2.5}$$

We shall construct several VPs dual to (2.5) in some of the variables, assuming for clarity that the dissipative mechanisms are unconnected. To construct the VP in terms of the variables $\mathbf{I}_m$, $\mathbf{P}_{mk}$, $\mathbf{I}_q$ we shall begin with the variation $\delta B_3^\circ(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q)$, where

$$\mathbf{I}_m=(\mathbf{J}_1,\ldots,\mathbf{J}_m),\quad \mathbf{P}_{mk}=(-\nabla p_{m+1},\ldots,-\nabla p_k),\quad \mathbf{I}_q=(Q_{21},\ldots,Q_{k1},Q_{32},\ldots,Q_{k2},\ldots,Q_{k,k-1})$$

Introducing the notation

$$\Psi_3(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q)=\Psi_m(\mathbf{I}_m)-\Phi_{mk}(\mathbf{P}_{mk})+\Psi_q(\mathbf{I}_q)$$

$$\Psi_m(\mathbf{I}_m)=\sum_{i=1}^{m}\Psi_i,\quad \Phi_{mk}(\mathbf{P}_{mk})=\sum_{i=m+1}^{k}\Phi_i,\quad \Psi_q(\mathbf{I}_q)=\sum_{j>i}^{k}\Psi_{ji}$$

after reduction we obtain

$$\delta B_3^\circ(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q)=\int_\Omega\left(\delta\Psi_3(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q)+\sum_{i=1}^{m}\nabla p_i^\circ\delta\mathbf{J}_i-\sum_{i=m+1}^{k}\mathbf{J}_i^\circ\delta\nabla p_i-\right.$$

$$\left.-\sum_{j>i}^{k}(p_j^\circ-p_i^\circ)\delta Q_{ji}\right)d\Omega=\int_\Omega\left(\delta\Psi_3(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q)+\sum_{i=1}^{m}\operatorname{div}(p_i^\circ\delta\mathbf{J}_i)-\right.$$

$$-\sum_{i=1}^{m}p_i^\circ\operatorname{div}\delta\mathbf{J}_i-\sum_{i=m+1}^{k}\operatorname{div}(\mathbf{J}_i^\circ\delta p_i)+\sum_{i=m+1}^{k}\operatorname{div}\mathbf{J}_i^\circ\delta p_i+\sum_{i=1}^{m}p_i^\circ\sum_{j=1}^{k}\delta Q_{ji}+$$

$$+\sum_{i=m+1}^{k} p_i^\circ \sum_{j=1}^{k} \delta Q_{ji}\Bigg)d\Omega = \int_\Omega \Bigg(\delta\Psi_3(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q) - \sum_{i=1}^{m} p_i^\circ \delta\left(\operatorname{div}\mathbf{J}_i - \sum_{j=1}^{k} Q_{ji}\right) +$$

$$+\sum_{i=m+1}^{k}\left(\operatorname{div}\mathbf{J}_i^\circ - \sum_{j=1}^{k} Q_{ji}^\circ\right)\delta p_i + \sum_{i=m+1}^{k}\left(\delta p_i \sum_{j=1}^{k} Q_{ji}^\circ + p_i^\circ \sum_{j=1}^{k}\delta Q_{ji}\right)\Bigg)d\Omega +$$

$$+\sum_{i=1}^{m}\int_\Gamma p_i^\circ \delta J_{in} d\Gamma - \sum_{i=m+1}^{k}\int_\Gamma J_{in}^\circ \delta p_i d\Gamma$$

Using the equalities

$$\operatorname{div}\mathbf{J}_i^\circ - \sum_{j=1}^{k} Q_{ji}^\circ = \sigma_i^*, \quad \delta p_i \sum_{j=1}^{k} Q_{ji}^\circ + p_i^\circ \sum_{j=1}^{k}\delta Q_{ji} = \delta\left(p_i \sum_{j=1}^{k} Q_{ji}\right)$$

which preserve the solution of the problem, we obtain the variational equation

$$\int_\Omega \Bigg(\delta\Psi_3(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q) - \sum_{i=1}^{m} p_i^\circ \delta\left(\operatorname{div}\mathbf{J}_i - \sum_{j=1}^{k} Q_{ji}\right) + \sum_{i=m+1}^{k}\sigma_i^*\delta p_i +$$
$$+\sum_{i=m+1}^{k}\delta\left(p_i \sum_{j=1}^{k} Q_{ji}\right)\Bigg)d\Omega + \sum_{i=1}^{m}\int_\Gamma p_i^\circ \delta J_{in} d\Gamma - \sum_{i=m+1}^{k}\int_\Gamma J_{in}^\circ \delta p_i d\Gamma = 0 \tag{2.6}$$

Under the constraints on the variables

$$\operatorname{div}\mathbf{J}_i = \sum_{j=1}^{k} Q_{ji} + \sigma_i^*, \quad i = 1,2,\ldots,m \tag{2.7}$$

$$J_{in} = J_{in}^\circ \quad \text{on} \quad \Gamma_{iq}, \quad i = 1,2,\ldots,m \tag{2.8}$$

$$p_i = p_i^\circ \quad \text{on} \quad \bar{\Gamma}_{ip}, \quad i = m+1, m+2,\ldots,k \tag{2.9}$$

the variational equation (2.6) implies the VP

$$\inf_{\mathbf{I}_m,\mathbf{I}_q \in (2.7),(2.8)} \ \sup_{\mathbf{p}_{mk}\in(2.9)} I_3(\mathbf{I}_m,\mathbf{p}_{mk},\mathbf{I}_q) \tag{2.10}$$

where

$$I_3(\mathbf{I}_m,\mathbf{p}_{mk},\mathbf{I}_q) = \int_\Omega \Bigg(\Psi_3(\mathbf{I}_m,\mathbf{P}_{mk},\mathbf{I}_q) + \sum_{i=m+1}^{k} p_i\left(\sigma_i^* + \sum_{j=1}^{k} Q_{ji}\right)\Bigg)d\Omega +$$

$$+\sum_{i=1}^{m}\int_{\Gamma_{ip}} p_i^\circ J_{in} d\Gamma - \sum_{i=m+1}^{k}\int_{\Gamma_{iq}} J_{in}^\circ p_i d\Gamma, \quad \mathbf{p}_{mk} = (p_{m+1},\ldots,p_k)$$

In a similar way we can construct other VPs. For example, the VPs in terms of $(\mathbf{J}_1,\ldots,\mathbf{J}_m, p_1,\ldots, p_k)$ and $(p_1,\ldots,p_k)$ subject to the conditions

$$p_i = p_i^\circ \quad \text{on} \quad \Gamma_{ip}, \quad i = 1,2,\ldots,k \tag{2.11}$$

have the form

$$\inf_{\mathbf{I}_m\in(2.8)} \ \sup_{\mathbf{p}\in(2.9)} I_4(\mathbf{I}_m,\mathbf{p}), \quad \inf_{\mathbf{p}\in(2.11)} I_2(\mathbf{p}) \tag{2.12}$$

where

$$I_4(\mathbf{I}_m,\mathbf{p}) = \int_\Omega \left(\Psi_m(\mathbf{I}_m) - \Phi_{mk}(\mathbf{P}_{mk}) - \Phi_q(\mathbf{P}_q) - \sum_{i=1}^{m} p_i \operatorname{div}\mathbf{J}_i + p_i\sigma_i^*\right)d\Omega +$$

$$+\sum_{i=1}^{m}\int_{\Gamma_{ip}} p_i^\circ J_{in} d\Gamma - \sum_{i=m+1}^{k}\int_{\Gamma_{iq}} J_{in}^\circ p_i d\Gamma$$

$$I_2(\mathbf{p}) = \int_\Omega (\Phi(\mathbf{P}) - p_i\sigma_i^*) d\Omega + \sum_{i=1}^{k}\int_{\Gamma_{iq}} J_{in}^\circ p_i d\Gamma$$

$$\Phi_q(\mathbf{P}_q) = \sum_{j>i}^{k} \Phi_{ji}, \quad \mathbf{P}_q = (p_2 - p_1, \ldots, p_k - p_1, p_3 - p_2, \ldots, p_k - p_{k-1}), \quad \mathbf{p} = (p_1, \ldots, p_k)$$

The VP in terms of $\mathbf{p} = (p_1, \ldots, p_k)$ is well known [5]. Here $\mathbf{P} = \mathbf{P}(\mathbf{b})$, $\mathbf{b} = \mathbf{p}$. It can be shown by direct verification that the VPs (2.5), (2.10) and (2.12) are equivalent to the solution of the system of equations (2.1) and (2.2) with boundary conditions (2.4) and (2.11) and the equalities

$$\inf_{\mathbf{I}\in(2.1),(2.4)} I_1(\mathbf{I}) = \sup_{\mathbf{p}\in(2.11)} [-I_2(\mathbf{p})] = \inf_{\mathbf{I}_m\in(2.8)} \sup_{\mathbf{p}\in(2.9)} I_4(\mathbf{I}_m, \mathbf{p}) =$$
$$= \inf_{\mathbf{I}_m, \mathbf{I}_q\in(2.7),(2.8)} \sup_{\mathbf{p}_{mk}\in(2.9)} I_3(\mathbf{I}_m, \mathbf{p}_{mk}, \mathbf{I}_q)$$

hold. When choosing the VP it is not required to know the boundary conditions. The combinations of unknown variables given on the boundary for which there is a solution of the problem can easily be found by analysing the boundary integrals. To obtain $I_2(\mathbf{p})$ without using the boundary conditions one should take the boundary integrals along the whole boundary Γ

$$\sum_{i=1}^{k}\int_{\Gamma} J_{in}^\circ p_i d\Gamma$$

which corresponds to the boundary conditions $J_{in} = J_{in}^\circ$ on Γ_q $(i = 1, 2, \ldots, k)$.

To use the boundary conditions (2.4) and (2.11) the variables p_i subject to variation must satisfy (2.11). Then

$$\sum_{i=1}^{k}\int_{\Gamma} J_{in}^\circ p_i d\Gamma = \sum_{i=1}^{k}\int_{\Gamma_{iq}} J_{in}^\circ p_i d\Gamma + \sum_{i=1}^{k}\int_{\Gamma_{ip}} J_{in}^\circ p_i^\circ d\Gamma$$

and the integrals over Γ_{ip}, which are constant, can be omitted. If $p_i = p_c$ on Γ_{iq}, $\Gamma_{iq} = \Gamma_q$ $(i = 1, 2, \ldots, k)$ then to determine the solution it suffices to specify the normal component $J_n^\circ = J_{1n}^\circ + \ldots + J_{kn}^\circ$

$$\sum_{i=1}^{k}\int_{\Gamma_{iq}} J_{in}^\circ p_i d\Gamma = \int_{\Gamma_q} J_n^\circ p_c d\Gamma$$

If $p_c = \text{const}$ on Γ_q, then to determine the solution it suffices to know the total divergence G° on Γ_q

$$\int_{\Gamma_q} J_n^\circ p_c d\Gamma = G^\circ p_c \quad \left(G^\circ = \int_{\Gamma_q} J_n^\circ d\Gamma\right)$$

where p_c is unknown, even if constant on Γ_q.

3. VARIATIONAL PRINCIPLES OF SEEPAGE CONSOLIDATION FOR A DEFORMABLE MEDIUM OF COMPLEX RHEOLOGY

We write the system of equations of seepage consolidation [6] in the form

$$\sigma_{ij,j}^f - p_{,i} = 0, \quad \operatorname{div}\mathbf{q} + \operatorname{div}\dot{\mathbf{u}} = 0 \tag{3.1}$$

$$-\mathbf{q} = \partial\Phi_p(\nabla p)/\partial\nabla p \quad \text{if} \quad -\nabla p = \partial\Psi_q(\mathbf{p})/\partial\mathbf{q} \tag{3.2}$$

$$\sigma_{ij}^f = F_{ij}(\varepsilon_{ij}, e_{ij}) \tag{3.3}$$

where (3.1) are the balance equations, (3.2) and (3.3) are the constitutive relations for the fluid and solid phases, $\Psi_q(q)$ and $\Phi_p(\nabla p)$ are the dissipative and adjoint dissipative potentials for the fluid phase, $\mathbf{q}$ is the rate of seepage, p is the pressure, σ^f_{ij} are the components of the effective stress tensor, u_i are the components of the displacement vector $\mathbf{u}$ and $\varepsilon_{ij} = 1/2(u_{i,j} + u_{j,i})$ are the components of the strain tensor $\boldsymbol{\epsilon}$, $e_{ij} = \dot{\varepsilon}_{ij}$. The solid phase can be modelled by a viscous element, which is, in turn, connected with a Kelvin–Voight element [4]. In the viscous element the dissipative mechanism is defined by a potential $\Psi_1(\mathbf{e}_1)$ and in the Kelvin–Voight element by $\Psi_2(\mathbf{e}_2)$. The mechanism of potential accumulation is given by $W_3(\boldsymbol{\epsilon}_3)$. Thus, the constitutive relation (3.3) has the form

$$\sigma_{1ij} = \partial\Psi(\mathbf{e}_1)/\partial e_{1ij}, \quad \sigma_{2ij} = \partial\Psi_2(\mathbf{e}_2)/\partial e_{2ij}, \quad \sigma_{3ij} = \partial W_3(\boldsymbol{\epsilon}_3)/\partial \varepsilon_{3ij} \tag{3.4}$$

$$\sigma^f_{ij} = \sigma_{1ij} = \sigma_{2ij} + \sigma_{3ij}, \quad e_{ij} = e_{1ij} + e_{2ij}, \quad e_{2ij} = e_{3ij}$$

We will construct the VP in terms of $\mathbf{u}$ and p, which are the usual variables used in numerical solutions of consolidation problems. Considering the minimum rates of energy dissipation and accumulation, we obtain the following variational equation corresponding to (1.2)

$$\delta \int_\Omega (\Psi_1(\mathbf{e}_1) + \Psi_2(\mathbf{e}_2) + \dot{W}_3(\boldsymbol{\epsilon}_3) + \Psi_q(\mathbf{q}) - \sum_{k=1}^{3} \sigma^\circ_{kij} e_{kij} + \nabla p^\circ \mathbf{q})\, d\Omega = 0 \tag{3.5}$$

where $\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, \dot{\mathbf{u}}^3, \mathbf{q}$ are the variables subject to variation, $\dot{W}_3(\boldsymbol{\epsilon}_3) = (\partial W_3(\boldsymbol{\epsilon}_3)/\partial\varepsilon_{3ij})\dot{\varepsilon}_{3ij}$ and $\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3$ being the displacement vectors for the elements 1–3, respectively. It follows that the variational equation for constructing the VP in terms of $\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, \dot{\mathbf{u}}^3$ and p taking the equality $\dot{\mathbf{u}}^2 = \dot{\mathbf{u}}^3$ into account has the form

$$\delta I(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) = \delta \int_\Omega \left(\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - \sum_{k=1}^{3} \sigma^\circ_{kij} e_{kij} - \mathbf{q}^\circ \nabla \mathbf{p} \right) d\Omega = 0 \tag{3.6}$$

where

$$\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) = \Psi_1(\mathbf{e}_1) + \Psi_2(\mathbf{e}_2) + \dot{W}_3(\boldsymbol{\epsilon}_2) - \Phi_p(\nabla p)$$

It can be seen that Eqs (3.5) and (3.6) are equivalent to the constitutive relations (3.2) and (3.3). After reduction we find from (3.6) that

$$\begin{aligned}
\delta I(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) &= \int_\Omega (\delta\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - \sigma^{f\circ}_{ij}\delta e_{1ij} - (\sigma^\circ_{2ij} + \sigma^\circ_{3ij})\delta e_{2ij} - q^\circ_i \delta p_{,i})\, d\Omega = \\
&= \int_\Omega (\delta\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - \sigma^{f\circ}_{ij}\delta e_{ij} - q^\circ_i \delta p_{,i})\, d\Omega = \\
&= \int_\Omega (\delta\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - \sigma^{f\circ}_{ij}\delta \dot{u}_{i,j} + p^\circ \delta\dot{u}_{i,i} - p^\circ\delta\dot{u}_{i,i} - \dot{u}^\circ_{i,i}\delta p + \dot{u}^\circ_{i,i}\delta p + q^\circ_{i,i}\delta p)\, d\Omega - \\
&- \int_\Gamma q^\circ_n \delta p\, d\Gamma = \int_\Gamma (\delta\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - (\sigma^{f\circ}_{ij} - p^\circ\delta_{ij})\delta\dot{u}_{i,j} - (p^\circ\delta\dot{u}_{i,i} + \delta p \dot{u}^\circ_{i,i}) + \\
&+ (u^\circ_{i,i} + q^\circ_{i,i})\delta p)\, d\Omega - \int_\Gamma q^\circ_n \delta p\, d\Gamma = \int_\Omega (\delta\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) + (\sigma^{f\circ}_{ij} - p^\circ\delta_{ij})_{,j}\delta\dot{u}_i - \\
&- (p^\circ \delta \operatorname{div}\dot{\mathbf{u}} + \delta p \operatorname{div}\dot{\mathbf{u}}^\circ) + (\operatorname{div}\dot{\mathbf{u}}^\circ + \operatorname{div}\mathbf{q}^\circ)\delta p)\, d\Omega - \int_\Gamma \Pi^\circ_i \delta\dot{u}_i\, d\Gamma - \int_\Gamma q^\circ_n \delta p\, d\Gamma, \\
\Pi_i &= (\sigma^f_{ij} - p\delta_{ij})n_j
\end{aligned}$$

Using the substitutions and variations of the functional

$$(\sigma^{f\circ}_{ij} - p^\circ\delta_{ij})_{,j} = 0, \quad \operatorname{div}\dot{\mathbf{u}}^\circ + \operatorname{div}\mathbf{q}^\circ = 0$$

$$p^\circ \delta \operatorname{div}\dot{\mathbf{u}} + \delta p \operatorname{div}\dot{\mathbf{u}}^\circ = \delta(p \operatorname{div}\dot{\mathbf{u}})$$

which preserve the solution $\dot{u}^\circ, p^\circ$ of the problem in (3.6), we obtain the variational equation

$$\delta \int_{\Omega} (\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - p\,\mathrm{div}(\dot{\mathbf{u}}^1 + \dot{\mathbf{u}}^2))d\Omega - \int_{\Gamma} \Pi_i^\circ \delta(\dot{u}_i^1 + \dot{u}_i^2)d\Gamma - \int_{\Gamma} q_n^\circ \delta p d\Gamma = 0 \tag{3.7}$$

Equation (3.7) implies the VP

$$\inf_{\dot{\mathbf{u}}^1,\, \dot{\mathbf{u}}^2,} \sup_{p} I(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2; p) \tag{3.8}$$

which is equivalent to solving the system of equations (3.1), (3.2) and (3.4) with boundary conditions $\Pi_i = \Pi^\circ_i$, $q_n = q^\circ_n$ on Γ, where

$$I(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) = \int_{\Omega} (\Psi(\dot{\mathbf{u}}^1, \dot{\mathbf{u}}^2, p) - p\,\mathrm{div}(\dot{\mathbf{u}}^1 + \dot{\mathbf{u}}^2))d\Omega - \int_{\Gamma} \Pi_i^\circ (\dot{u}_i^1 + \dot{u}_i^2)d\Gamma - \int_{\Gamma} q_n^\circ p d\Gamma$$

The deformation field $\epsilon_2 = \epsilon_2(t)$ in the VP (3.8) defines the accumulation of elastic energy at time t. The VP (3.4) holds whenever the potentials $\Psi_1(\mathbf{e}_1)$, $\Psi_2(\mathbf{e}_2)$ are non-differentiable and characterize the motion of viscoplastic and rigid-plastic media [7]. In the same way as in the previous computation one can construct the VPs for other choices of variables and determine various combinations of admissible boundary conditions.

Note that other VPs [8–12]†, which were obtained by different methods, can be derived using the above scheme. A similar scheme was used to construct VPs in seepage and consolidation theories in [13–15].‡

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†See also: KOSTERIN, A. V., Variational principle of seepage consolidation. Kazanskii Univ., Kazan, 1986. Deposited at VINITI 16.12.86, N8598-B.

‡See also: MAZUROV, P. A., Variational approach in the theory of seepage consolidation and the theory of two-phase seepage. Kazan. Fiz.-Tekhn. Institut Kazan. Fil. Akad. Nauk SSSR, Kazan, 1989. Deposited at VINITI 20.04.89, N2586-B.